stichting
mathematisch
centrum

MC

AFDELING NUMERIEKE WISKUNDE NW 22/75 AUGUST

P.J. VAN DER HOUWEN

TWO-LEVEL DIFFERENCE SCHEMES WITH VARYING MESH SIZES FOR THE SHALLOW WATER EQUATIONS

2e boerhaavestraat 49 amsterdam

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

Printed at the Mathematical Centre, 49, 2e Boerhaavestraat, Amsterdam.

The Mathematical Centre, founded the 11-th of February 1946, is a non-profit institution aiming at the promotion of pure mathematics and its applications. It is sponsored by the Netherlands Government through the Netherlands Organization for the Advancement of Pure Research (Z.W.O), by the Municipality of Amsterdam, by the University of Amsterdam, by the Free University at Amsterdam, and by industries.

AMS(MOS) subject classification scheme (1970): 65M05, 10, 20, 86A05

Two-level difference schemes with varying mesh sizes for the shallow water equations

by

P.J. van der Houwen

ABSTRACT

The purpose of this investigation is threefold: it gives a survey of known two-level difference schemes for the shallow water equations, it describes how non-uniform mesh sizes can be introduced into these schemes and finally, it proposes a new, explicit two-level scheme which remains stable for vanishing bottom friction and which has a more accurate discretization of the time-derivative than existing, explicit methods.

KEY WORDS & PHRASES: *Difference schemes, shallow water problems, varying mesh sizes*

CONTENTS

INTRODUCTION

In this report we propose a numerical scheme for two-dimensional water-wave propagation in shallow seas. This renewed research on shallow water problems has been started at the Mathematical Centre because of some recent quotions about the computation of the water elevation in the Adriatic Sea and the estuary near the Port of Calcutta. It turned out that there is a great demand for a difference scheme with the following properties:

(1) it should be explicit
(2) it should remain stable for vanishing bottom friction
(3) varying mesh sizes should be easily introduced
(4) the errors due to discretization of the space coordinates and time coordinate should be of the same order.

Before we construct such a scheme, we first review the several computational models proposed in the literature; in this report we confine our considerations to two-level schemes. In a forthcoming paper we intend to consider higher-level schemes.

None of the available schemes have the possibility to adapt variable mesh sizes, therefore, our purpose is to modify the difference schemes in such a way that variable mesh sizes can be introduced by providing some geometrical data about the grid. By adapting the stability conditions to the non-uniform meshes, we shall also derive a criterion how to choose the grid as economical as possible (with respect to computational effort) for a given depth function. Finally, we propose a new difference scheme which satisfies criteria (1), (2) and (3), and criterion (4) when non-linear terms are omitted in the mathematical model. This scheme has been tested for the North Sea model with a special, non-uniform rectangular grid. The results indicate a considerable improvement in efficiency when compared with the uniform-grid methods. In the near future we will report numerical results of our experiments.

1. THE MATHEMATICAL MODEL

The equations governing the water motion in a shallow sea are given by (cf. HANSEN [1956])

$$\frac{\partial U}{\partial t} = -\left[\lambda + \frac{\partial}{\partial x} U\right] U + \left[\omega - \frac{\partial}{\partial y} U\right] V - g \frac{\partial}{\partial x} Z + F_x$$

(1.1) $$\frac{\partial V}{\partial t} = -\left[\omega + \frac{\partial}{\partial x} V\right] U - \left[\lambda + \frac{\partial}{\partial y} V\right] V - g \frac{\partial}{\partial y} Z + F_y ,$$

$$\frac{\partial Z}{\partial t} = -\frac{\partial}{\partial x} [(h+Z)U] - \frac{\partial}{\partial y} [(h+Z)V]$$

where U,V are the vertically averaged velocity components in the x-and y-directions, respectively

Z is the water elevation,

λ is the bottom friction coefficient which may be expressed as (cf. DRONKERS [1964])

(1.2) $$\lambda = g \frac{\sqrt{U^2+V^2}}{C^2(h+Z)} ,$$ C_1 coefficient of De Chezy

ω is the Coriolis parameter given by (cf. PROUDMAN [1953])

(1.3) $\omega = 2\alpha \sin \phi$, α: angular rotation of the earth, ϕ: latitude of the location,

g is the acceleration of gravity,

F_x, F_y are the forcing functions of wind stress and barometric pressures in the x- and y- directions, respectively,

h is the depth of the undisturbed sea.

Equations (1.1) represent a *total hyperbolic* system (cf. VAN DER HOUWEN [1968]).

The boundary conditions are

(1.4) $Z = 0$ along Γ_{oc} ,

(1.5) $\vec{W} = (U,V)^T$ parallel to the coast along Γ_{oc} ,

where Γ_{oc} and Γ_c represent the ocean border and coast line, respectively. Condition (1.5) may be written as (cf. VAN DER HOUWEN [1968]).

(1.5') $$\frac{\partial}{\partial t}\vec{W} = -[(\vec{W}\cdot\nabla) + \lambda]\vec{W} - g(\vec{C}\cdot\nabla Z)\vec{C} + (\vec{C}\cdot\vec{F})\vec{C},$$

where $\nabla = \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right)^T$, $\vec{F} = (F_x,F_y)^T$, $(\cdot)$ denotes the usual inner-product and where $C = (C_x,C_y)$ is a unit vector tangential (in the positive sense) to the coast.

When the initial state of the sea is given, equations (1.1), (1.4) and (1.5) completely determine the subsequent motion of the sea.

2. TRANSFORMATION OF THE SPACE COORDINATES

In the numerical solution of system (1.1) by difference methods, it often is desirable to use a non-uniform grid in the (x,y)-plane. However, the construction of difference quotients at non-uniform meshes is rather complicated; hence, we prefer to introduce new coordinates X and Y, such that the curves

(2.1) $$X(x,y) = j\Delta \qquad Y(x,y) = k\Delta \qquad j,k \text{ integers, } \Delta \text{ constant,}$$

determine the grid desired in the (x,y)-plane. In the (X,Y)-plane we then have a square grid with a width Δ.

The price we have to pay is, of course, a more complicated system of differential equations, as the operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are to be replaced by

(2.2) $$\frac{\partial X}{\partial x}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial}{\partial Y} \text{ and } \frac{\partial X}{\partial y}\frac{\partial}{\partial X} + \frac{\partial Y}{\partial y}\frac{\partial}{\partial Y} ,$$

respectively.

When substituted into (1.1) we see that we do not need the functions X and Y themselves; only the partial derivatives are needed in order to carry out the transformation. We shall give a sketch how these derivatives can be derived from a given net in the (x,y)-plane (see figure 2.1).

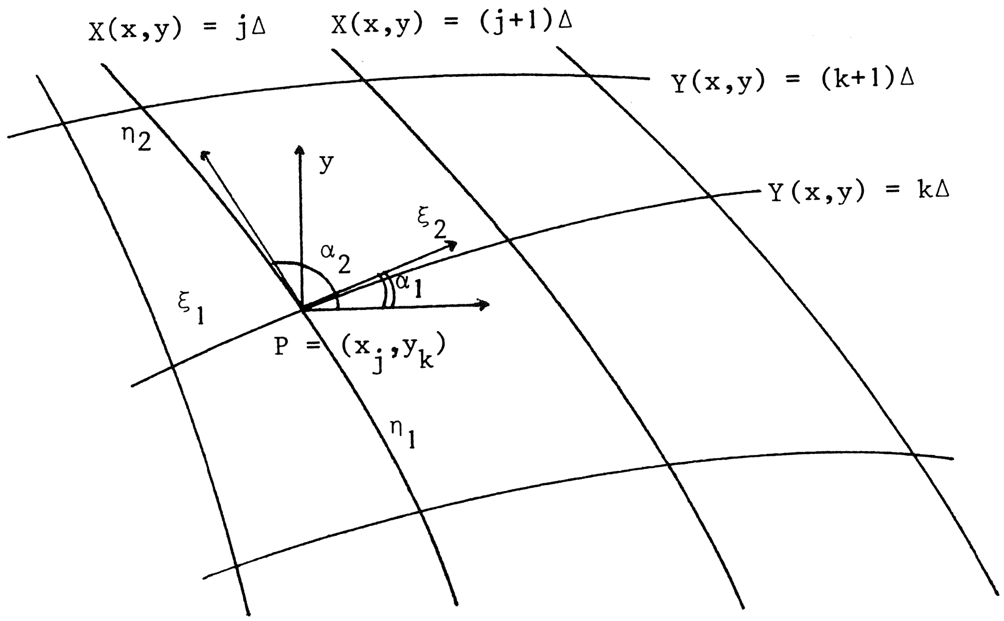

Fig. 2.1. Curvilinear net in the (x,y)-plane

Let the lines of tangency at the point P = (x_j, y_k) of the given net make angles α_1 and α_2 with the positive x-axis and let the neighbouring net points be at distance ξ_1, ξ_2, η_1 and η_2 .

Let D_α F denote the derivative of a function F in the direction of a vector which makes an angle α with the positive x-axis. Obviously, the functions X and Y satisfy the conditions

(2.3) $$D_{\alpha_2} X = D_{\alpha_1} Y = 0$$

at every grid point of the given net. In order to determine all partial derivatives of X and Y we need two more conditions on X and Y. Let us choose X and Y such that

(2.4) $$D_{\alpha_1} X = \frac{2\Delta}{\xi_1+\xi_2} \,, \quad D_{\alpha_2} Y = \frac{2\Delta}{\eta_1+\eta_2}$$

holds at every grid point P. Then we obtain for the partial derivatives with respect to x and y the equations

(2.5) $$\begin{aligned} D_{\alpha_2} X &= \cos\alpha_2 \frac{\partial X}{\partial x} + \sin\alpha_2 \frac{\partial X}{\partial y} = 0, \\ D_{\alpha_1} X &= \cos\alpha_1 \frac{\partial X}{\partial x} + \sin\alpha_1 \frac{\partial X}{\partial y} = \frac{2\Delta}{\xi_1+\xi_2} \,, \\ D_{\alpha_1} Y &= \cos\alpha_1 \frac{\partial Y}{\partial x} + \sin\alpha_1 \frac{\partial Y}{\partial y} = 0, \\ D_{\alpha_2} Y &= \cos\alpha_2 \frac{\partial Y}{\partial x} + \sin\alpha_2 \frac{\partial Y}{\partial y} = \frac{2\Delta}{\eta_1+\eta_2} \,. \end{aligned}$$

It is easily verified that

(2.6) $$\frac{\partial X}{\partial x} = \frac{\Delta}{\bar{\xi}} \frac{\sin\alpha_2}{\sin(\alpha_2-\alpha_1)} \,, \quad \frac{\partial X}{\partial y} = \frac{\Delta}{\bar{\xi}} \cdot \frac{\cos\alpha_2}{\sin(\alpha_1-\alpha_2)}$$

and

(2.7) $$\frac{\partial Y}{\partial x} = \frac{\Delta}{\bar{\xi}} \frac{\sin\alpha_1}{\sin(\alpha_1-\alpha_2)} \,, \quad \frac{\partial Y}{\partial y} = \frac{\Delta}{\bar{\eta}} \frac{\cos\alpha_1}{\sin(\alpha_2-\alpha_1)}$$

where we have put

$$\frac{\xi_1+\xi_2}{2} = \bar{\xi} \,, \quad \frac{\eta_1+\eta_2}{2} = \bar{\eta} \,.$$

We remark that the coast conditions become particularly simple when we choose the net lines along the coast and when we choose $\alpha_2-\alpha_1 = \pm \frac{\pi}{2}$ at coastal grid points. In the (X,Y)-system we then have either U = 0 or V = 0 at the coast.

3. DISCRETIZATION OF THE SPACE-DERIVATIVES

In the first papers on the computation of shallow water problems, a drastically simplified version was used for the mathematical model described in section 1. For instance, the convective-inertia term $(\vec{W}\cdot\nabla)\vec{W}$ was neglected, the coefficient of friction λ was taken to be constant, and the elevation Z was neglected with repsect to the depth h. When these simplifications are introduced into system (1.1), we obtain the *linearized* system

$$\frac{\partial U}{\partial t} = - \lambda U + \omega V - g \frac{\partial}{\partial x} Z + F_x,$$

$$(3.1) \qquad \frac{\partial V}{\partial t} = - \omega U - \lambda V - g \frac{\partial}{\partial y} Z + F_y,$$

$$\frac{\partial Z}{\partial t} = - h \frac{\partial}{\partial x} U - h \frac{\partial}{\partial y} V.$$

This system is usually taken as a starting point; when a discrete approximation to (3.1) is established one tries to fit in the *non-linear* terms. In this section we discuss the discretization of the right hand side of system (3.1). Several methods of discretization have been proposed in the literature, which mainly differ by the fact that the stream and elevation fields are calculated in *different* grid points (*space-staggered grids*).

3.1. THE FISCHER GRID

When in system (3.1) the differential operators $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ (or alternatively the differential operators $\frac{\partial}{\partial X}$ and $\frac{\partial}{\partial Y}$ when (2.2) is used) are replaced by the usual central difference operators, it is readily seen that at points where U and V are needed, the Z-component is *not* needed and vice versa. This suggests a space-staggered grid as shown in figure 3.1

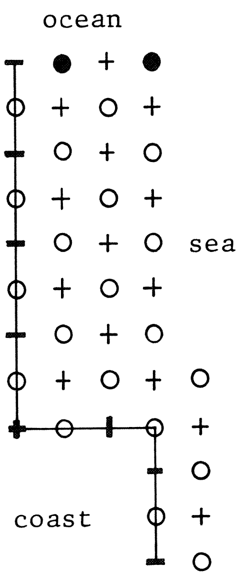

fig. 3.1 Grid used by
FISCHER [1959]
and SIELECKI [1967]

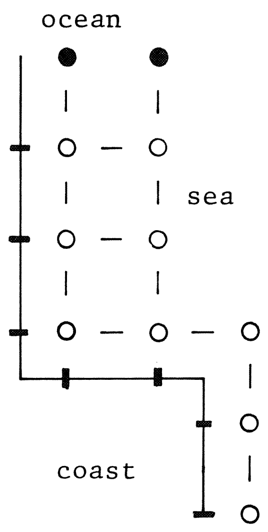

fig. 3.2. Grid used by
PLATZMAN [1959],
HANSEN [1961] and
LEENDERTSE [1967]

In this and following figures we use the notation:

(3.2)

- : grid point where U is computed,
| : grid point where V is computed,
o : grid point where Z is computed,
= : U = 0, ‖ : V = 0, • : Z = 0.

The grid specified in figure 3.1 was actually used by FISCHER [1959] and Miss SIELECKI [1967]. Along the ocean border, Fischer prescribes both Z, U and V. Along the coast the non-zero component of the stream and the elevation Z is handled with asymmetric differences. When the nonlinear terms are again added to system (3.1), we need Z at the stream points. These values can be provided by averaging the elevation values of the four neighbouring elevation points; at the coastal points only two neighbouring elevation points can be averaged. Furthermore, the convective-inertia terms may be discretized at internal streampoints by using average central differences. At coastal stream points one usually neglects convective-inertia.

3.2. THE PLATZMAN GRID

In the same year as Fischer, Platzman proposed another space-staggered grid (see figure 3.2) which has the advantage that no asymmetric difference operators are involved in order to represent the coast conditions. But, since U and V are computed at different points, an averaging process is needed to provide U at the V-points and V at the U-points. A more complicated point is, however, the introduction of new coordinates X and Y; in particular, close to the coast we get lengthy, asymmetric difference formulas, unless the function X only depends on x and Y only on y. When non-linear terms are taken into account, we also have to provide Z at the U- and V-points. As in the Fisher case one neglects convective-inertia terms at the coast.
The Platzman grid has been used by HANSEN [1961] and LEENDERTSE [1967].

3.3. THE LAUWERIER-DAMSTÉ GRID

In practice, the Fischer grid turned out to be not completely satisfying; in order to remove small oscillations in the stream and elevation fields, one had to use smoothing operators (cf. FISCHER [1959]). LAUWERIER & DAMSTÉ [1961] discovered that by rotating the Fischer grid 45^0 degrees (see figure 3.3) and by using averaged central differences, automatically some kind of smoothing is obtained. Boundary conditions have to be discretized by asymmetric difference quotients. A detailed treatment of boundary conditions in the Lauwerier-Damstě grid has been given by HEAPS [1969]. Introduction of non-linear terms offers no more difficulties than in the Fischer case.

3.4. COMBINATION OF THE PLATZMAN AND LAUWERIER-DAMSTÉ GRIDS

In figure 3.4 an attempt is presented to combine the advantage of the Platzman grid (easy boundary conditions) with the advantage of the

Lauwerier-Damsté grid (smoothing of the elevation field). But, due to the Platzman configuration along the coast, only transformation functions of the form $X = X(x)$ and $Y = Y(y)$ maintain the simplicity of the boundary conditions.

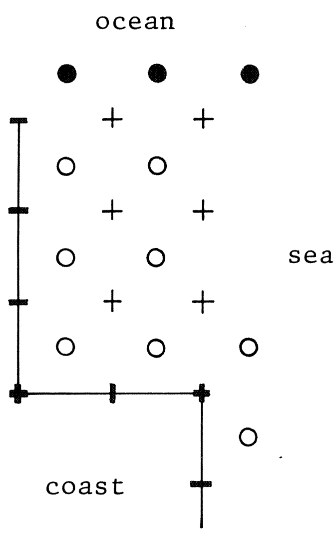

fig. 3.3. Grid used by LAUWERIER & DAMSTÉ [1961], VAN DER HOUWEN [1966] and HEAPS [1969]

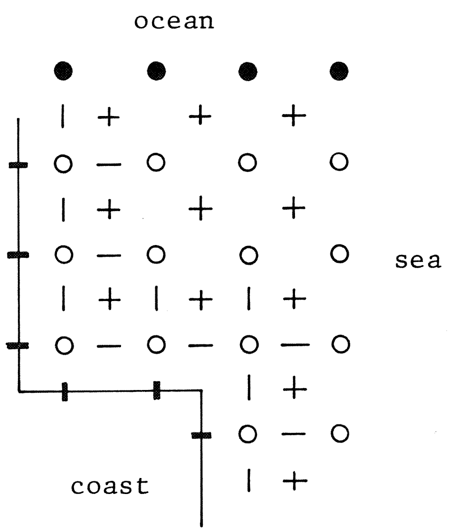

fig. 3.4. Grid used by VAN DER HOUWEN [1969, unpublished]

3.5. THE METHOD OF LINES

Having chosen a particular grid, we may discretize the operators $\partial/\partial x$ and $\partial/\partial y$ with respect to this grid. We denote the discretized differential operators by (compare (2.2))

$$[\frac{\partial}{\partial x}] = [\frac{\partial X}{\partial x}][\frac{\partial}{\partial X}] + [\frac{\partial Y}{\partial x}][\frac{\partial}{\partial Y}] \text{ and } [\frac{\partial}{\partial y}] = [\frac{\partial X}{\partial y}][\frac{\partial}{\partial X}] + [\frac{\partial Y}{\partial y}][\frac{\partial}{\partial Y}] ,$$

where $[\frac{\partial X}{\partial x}],\ldots$ represent the functions $\frac{\partial X}{\partial x},\ldots$ when restricted to the grid points. Furthermore, the stream (U,V) and the elevation Z, when restricted to the grid points, will be denoted by $(\vec{u},\vec{v})$ and $\vec{z}$. Note that the number of components of these vectors equal the number of grid points used to represent the sea and its coasts. System (1.1) can now be approximated by a sent the sea and its coasts. System (1.1) can now be approximated by a *system of ordinary differential equations* (method of lines)

$$
(3.3)\qquad
\begin{aligned}
\frac{d\vec{u}}{dt} &= -\,(\vec{\lambda}+[\tfrac{\partial}{\partial x}]\vec{u})\vec{u} + [\vec{\omega} - [\tfrac{\partial}{\partial y}]\vec{u}]\vec{v} - g[\tfrac{\partial}{\partial x}]\vec{z} + \vec{f}_x,\\
\frac{d\vec{v}}{dt} &= -\,(\vec{\omega}+[\tfrac{\partial}{\partial x}]\vec{v})\vec{u} - [\vec{\lambda} + [\tfrac{\partial}{\partial y}]\vec{v}]\vec{v}] - g[\tfrac{\partial}{\partial y}]\vec{z} + \vec{f}_y,\\
\frac{d\vec{z}}{dt} &= -\,[\tfrac{\partial}{\partial x}][(\vec{h}+\vec{z})\vec{u}] - [\tfrac{\partial}{\partial y}][(\vec{h}+\vec{z})\vec{v}].
\end{aligned}
$$

Here, the multiplication of vectors is understood to be carried out componentwise. The vectors $\vec{\lambda}$, $\vec{\omega}$, $\vec{f}_x$, $\vec{f}_y$ and $\vec{h}$ denote the fields of friction coefficeints, Coriolis parameters, wind friction in x and y direction, and the depth field, respectively. Equations (3.3) hold at the internal net points. Along the coast, we have to discretize equation (1.5'). It is easily verified that we obtain

$$
(3.4)\qquad
\begin{aligned}
\frac{d\vec{u}}{dt} &= -\,(\vec{\lambda}+[\tfrac{\partial}{\partial x}]\vec{u})\vec{u} - ([\tfrac{\partial}{\partial y}]\vec{u})\vec{v} - g\vec{c}_x(\vec{c}_x[\tfrac{\partial}{\partial x}]+\vec{c}_y[\tfrac{\partial}{\partial y}])\vec{z} +\\
&\qquad + \vec{c}_x(\vec{c}_x\vec{f}_x+\vec{c}_y\vec{f}_y),\\
\frac{d\vec{v}}{dt} &= -\,([\tfrac{\partial}{\partial x}]\vec{v})\vec{u} - (\vec{\lambda}+[\tfrac{\partial}{\partial y}]\vec{v}) - g\vec{c}_y(\vec{c}_x[\tfrac{\partial}{\partial x}] + \vec{c}_y[\tfrac{\partial}{\partial y}])\,\vec{z} +\\
&\qquad + \vec{c}_y(\vec{c}_x\vec{f}_x+\vec{c}_y\vec{f}_y),\\
\frac{d\vec{z}}{dt} &= -\,[\tfrac{\partial}{\partial x}][(\vec{h}+\vec{z})\vec{u}] - [\tfrac{\partial}{\partial y}][(\vec{h}+\vec{z})\vec{v}]\ ,
\end{aligned}
$$

where $\vec{u}$, $\vec{v}$,... denote the U, V,... fields restricted to the grid points at the coast. Similarly, we have along the ocean the equations

$$
(3.5)\qquad
\begin{aligned}
\frac{d\vec{u}}{dt} &= -\,(\vec{\lambda}+[\tfrac{\partial}{\partial x}]\vec{u})\vec{u} + (\vec{\omega}-[\tfrac{\partial}{\partial y}]\vec{u})\vec{v} - g\,[\tfrac{\partial}{\partial x}]\,\vec{z} + \vec{f}_x\ ,\\
\frac{d\vec{v}}{dt} &= -\,(\vec{\omega}+[\tfrac{\partial}{\partial x}]\vec{v})\vec{u} - (\vec{\lambda}+[\tfrac{\partial}{\partial y}]\vec{v})\vec{v} - g\,[\tfrac{\partial}{\partial y}]\,\vec{z} + \vec{f}_y\ ,\\
\frac{d\vec{z}}{dt} &= 0.
\end{aligned}
$$

We recall that when the difference operators require values of $\vec{u},\vec{v},\vec{z},\ldots$ at points where these values are not calculated, one should provide these values by applying some averaging process.

4. DISCRETIZATION OF THE TIME-DERIVATIVE

Let us define the matrix

$$(4.1)\qquad D = -\begin{pmatrix} \vec{\lambda} + ([\frac{\partial}{\partial x}]\vec{u}) & -\vec{\omega} + ([\frac{\partial}{\partial y}]\vec{u}) & g[\frac{\partial}{\partial x}] \\ \vec{\omega} + ([\frac{\partial}{\partial x}]\vec{v}) & \vec{\lambda} + ([\frac{\partial}{\partial y}]\vec{v}) & g[\frac{\partial}{\partial y}] \\ [\frac{\partial}{\partial x}]\ (\vec{h}+\vec{z}) & [\frac{\partial}{\partial y}]\ (\vec{h}+\vec{z}) & 0 \end{pmatrix}$$

and the vectors $\vec{f} = (\vec{f}_x,\vec{f}_y,\vec{0})^T$ and $\vec{s} = (\vec{u},\vec{v},\vec{z})^T$. Then, by the convention that multiplication of vectors is carried out component-wise, we may write (3.3) in the form

$$(4.2)\qquad \frac{d}{dt}\,\vec{s} = D\vec{s} + \vec{f}.$$

Note that the elements of the third row and third column of the matrix D are difference operators; for instance, the element $[\partial/\partial x](\vec{h}+\vec{z})$ is an operator defined by

$$[\frac{\partial}{\partial x}](\vec{h}+\vec{z})\vec{u} = [\frac{\partial}{\partial x}].\ ((\vec{h}+\vec{z})\vec{u}).$$

The other elements of D may be interpreted as diagonal matrices, Furthermore, note that D depends on $\vec{s}$, unless the linearized form (3.1) is used. We then have

$$(4.1')\qquad D = -\begin{pmatrix} \vec{\lambda} & -\vec{\omega} & g[\frac{\partial}{\partial x}] \\ \vec{\omega} & \vec{\lambda} & g[\frac{\partial}{\partial y}] \\ h[\frac{\partial}{\partial x}] & h[\frac{\partial}{\partial y}] & 0 \end{pmatrix}$$

along the coast and ocean border we will also use representation (4.2), but since the state vector $\vec{s}$ is now described by equations (3.4) and (3.5), the matrix D and vector $\vec{f}$ should be accordingly changed.

In this section we give a survey of the most important single step methods known in the literature to solve system (4.2). Moreover, a new method, not yet published, is indicated.

4.1. METHOD OF EULER

The most simple method for integrating ordinary differential equations is Euler's method which reads for equation (4.2)

$$(4.3)\qquad \vec{s}_{n+1} = [I + \tau_n D(\vec{s}_n)]\,\vec{s}_n + \tau_n \vec{f}_n\ .$$

Here, $\vec{s}_n$, $\vec{s}_{n+1}$,... denote approximations to the solution $\vec{s}(t)$ at $t = t_n$, $t = t_{n+1}$,..., respectively, $\vec{f}_n = \vec{f}(t_n)$ and $\tau_n = t_{n+1} - t_n$. This method was investigated by LAUWERIER and DAMSTE [1961]. Its stability behaviour, however, is so poor that it is of no practical value. We return to this point in section 5.4.

4.2. STABILIZED RUNGE-KUTTA METHODS

In VAN DER HOUWEN [1971] a class of Runge-Kutta methods was developed which are suitable for the integration of the large systems of ordinary differential equations (1000 or more equations) arising from partial differential equations by discretizing the space-derivatives. A typical example of such a Runge-Kutta method which is adapted to equation (4.2) is given by

$$(4.4)\qquad \begin{aligned} \tilde{\vec{s}}_{n+1} &= \vec{s}_n + \tfrac{1}{2}\tau_n[D(\vec{s}_n)\vec{s}_n + \vec{f}(t_n)]\ , \\ \tilde{\tilde{\vec{s}}}_{n+1} &= \vec{s}_n + \tfrac{1}{2}\tau_n[D(\tilde{\vec{s}}_{n+1})\tilde{\vec{s}}_{n+1} + \vec{f}(t_n+\tfrac{1}{2}\tau_n)], \\ \vec{s}_{n+1} &= \vec{s}_n + \tau_n[D(\tilde{\tilde{\vec{s}}}_{n+1})\,\tilde{\tilde{\vec{s}}}_{n+1} + \vec{f}(t_{n+1})]. \end{aligned}$$

4.3. GENERALIZED EULER METHODS

The generalized Euler method is defined by

$$(4.5) \qquad \vec{s}_{n+1} = [I + \tau_n \Theta_n D(\vec{s}_n)]\vec{s}_n + \tau_n \Theta_n \vec{f}_n,$$

where Θ_n is a matrix operator depending on τ_n in such a way that

$$\Theta_n = I + O(\tau_n) \text{ as } \tau_n \to 0.$$

By writing Θ_n in the form

$$\Theta_n = [I - \tau_n E_n]^{-1} \quad ,$$

scheme (4.5) may be written in the alternative form

$$(4.5') \qquad [I - \tau_n E_n]\vec{s}_{n+1} = [I + \tau_n (D(\vec{s}_n) - E_n)]\vec{s}_n + \tau_n \vec{f}_n;$$

E_n is some matrix operator which makes the Euler method more or less *implicit*, that is E_n is a matrix between 0 (generating Euler's method) and $D(\vec{s}_{n+1})$ generating the backward Euler method). Usually, when large systems are involved, one chooses for E_n a triangular matrix in order to have effectively an explicit method. Most difference schemes used in the actual computation of shallow water problems can be written in the form (4.5) by specifying some operator Θ_n. We will discuss the schemes of Fischer, and Leendertse, and indicate some modifications.

4.3.1. FISCHER SCHEME

For linear models, that is D defined by (4.1'), FISCHER [1959] used a difference scheme of the form (4.5) with

$$(4.6) \qquad \Theta_n = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \tau_n \vec{h}[\frac{\partial}{\partial x}] & \tau_n \vec{h}[\frac{\partial}{\partial y}] & I \end{pmatrix}^{-1}$$

It should be remarked that HANSEN [1961] essentially used the same time-discretization as Fischer, however, in the Hansen scheme the state vectors $\vec{s}_n$ and $\vec{s}_{n+1}$ are understood to represent the vector $(\vec{u}_{n-\frac{1}{2}},\vec{v}_{n-\frac{1}{2}},\vec{z}_n)^T$ and $(\vec{u}_{n+\frac{1}{2}},\vec{v}_{n+\frac{1}{2}},\vec{z}_{n+1})^T$, respectively (time-staggered grid).

4.3.2. SIELECKI SCHEME

A slight modification of the Fischer scheme was proposed by Miss SIELECKI [1967]:

$$(4.7)\qquad \Theta_n = \begin{pmatrix} I & 0 & 0 \\ \tau_n\vec{\omega} & I & 0 \\ \tau_n\vec{h}[\frac{\partial}{\partial x}] & \tau_n\vec{h}[\frac{\partial}{\partial y}] & I \end{pmatrix}^{-1} .$$

4.3.3. FISCHER SCHEME WITH ARTIFICIAL VISCOSITY

In VAN DER HOUWEN [1968] Fischer's scheme is analysed when an artificial viscosity term is introduced. Again, restricting to linear models, the scheme is described by

$$(4.8)\qquad \Theta_n = \begin{pmatrix} I & 0 & 0 \\ 0 & I & 0 \\ \tau_n\vec{h}([\frac{\partial}{\partial x}] - q[\frac{\partial}{\partial y}]) & \tau_n\vec{h}(q[\frac{\partial}{\partial x}] + [\frac{\partial}{\partial y}]) & I \end{pmatrix}^{-1} ,$$

where q is a viscosity parameter.

4.3.4. LEENDERTSE SCHEME

When in the difference scheme of LEENDERTSE [1967] the non-linear terms are omitted, it can also be written in the form (4.5) with

$$(4.9)\qquad \Theta_n = \begin{pmatrix} I & 0 & \tau_n g[\frac{\partial}{\partial x}] \\ \tau_n\vec{\omega} & I + \tau_n\vec{\lambda} & 0 \\ \tau_n\vec{h}\frac{\partial}{\partial x} & 0 & I \end{pmatrix}^{-1} \quad \text{for n even}$$

and

$$(4.9)\qquad \Theta_n = \begin{pmatrix} I + \tau_n\vec{\lambda} & -\tau_n\vec{\omega} & 0 \\ 0 & I & \tau_n g[\frac{\partial}{\partial y}] \\ 0 & \tau_n\vec{h}[\frac{\partial}{\partial y}] & I \end{pmatrix} \quad \text{for n odd}$$

For non-linear models the Leendertse scheme becomes too complicated to present it in the above simple form. The interested reader is referred to the original paper.

4.3.5 SYMMETRIZED SCHEME

In order to reduce the error in the difference scheme due to the time-discretization we try to make the discretization of d/dt more symmetric, that is we try to make the time-discretization second order accurate (note that the schemes (4.3), (4.6), (4.7) and (4.8) are all first order accurate with respect to the time step). Let $\vec{s}_n$ and $\vec{s}_{n+1}$ denote the vector fields $(\vec{u}_{n-\frac{1}{2}},\vec{v}_{n-\frac{1}{2}},\vec{z}_n)$ and $(\vec{u}_{n+\frac{1}{2}},\vec{v}_{n+\frac{1}{2}},\vec{z}_{n+1})$, respectively. Then the operator

$$(4.10)\qquad \Theta_n = \begin{pmatrix} I + \frac{1}{2}\tau(\vec{\lambda}_x)_{n-\frac{1}{2}} & -\frac{1}{2}\tau(\vec{\omega}_x)_{n-\frac{1}{2}} & \vec{0} \\ \frac{1}{2}\tau(\vec{\omega}_y)_{n-\frac{1}{2}} & I + \frac{1}{2}\ (\vec{\lambda}_y)_{n-\frac{1}{2}} & \vec{0} \\ \tau[\frac{\partial}{\partial x}]\vec{\zeta}_n & \tau[\frac{\partial}{\partial y}]\vec{\zeta}_n & I \end{pmatrix}^{-1}$$

defines a method which is symmetric for linear models, provided that τ does not depend on the step number n. In order to simplify the formulas we have introduced into (4.10) the vectors:

$$\begin{aligned}
\vec{\lambda}_x &= \vec{\lambda} + [\frac{\partial}{\partial x}]\vec{u}\ , \\
\vec{\lambda}_y &= \vec{\lambda} + [\frac{\partial}{\partial y}]\vec{v}\ , \\
\vec{\omega}_x &= \vec{\omega} - [\frac{\partial}{\partial y}]\vec{u}\ , \\
\vec{\omega}_y &= \vec{\omega} + [\frac{\partial}{\partial x}]\vec{v}\ , \\
\vec{\zeta} &= \vec{h} + \vec{z}.
\end{aligned}$$

It may be helpful to present the scheme defined by (4.5) and the matrix Θ_n defined above in the following form:

$$\frac{\vec{u}_{n+\frac{1}{2}}-\vec{u}_{n-\frac{1}{2}}}{\tau} = -\tfrac{1}{2}(\vec{\lambda}_x)_{n-\frac{1}{2}}[\vec{u}_{n+\frac{1}{2}}+\vec{u}_{n-\frac{1}{2}}] + \tfrac{1}{2}(\vec{\omega}_x)_{n-\frac{1}{2}}[\vec{v}_{n+\frac{1}{2}}+\vec{v}_{n-\frac{1}{2}}] - g[\frac{\partial}{\partial x}]\vec{z}_n + (\vec{f}_x)_n,$$

$$(4.11) \qquad \frac{v_{n+\frac{1}{2}}-v_{n-\frac{1}{2}}}{\tau} = -\tfrac{1}{2}(\vec{\omega}_y)_{n-\frac{1}{2}}[\vec{u}_{n+\frac{1}{2}}+\vec{u}_{n-\frac{1}{2}}] - \tfrac{1}{2}(\vec{\lambda}_y)_{n-\frac{1}{2}}[\vec{v}_{n+\frac{1}{2}}+\vec{v}_{n-\frac{1}{2}}] - g[\frac{\partial}{\partial y}]\vec{z}_n + (\vec{f}_y)_n,$$

$$\frac{\vec{z}_{n+1}-\vec{z}_n}{\tau} = -[\frac{\partial}{\partial x}](\vec{\zeta}_n\vec{u}_{n+\frac{1}{2}}) - [\frac{\partial}{\partial y}](\vec{\zeta}_n\vec{v}_{n+\frac{1}{2}}) .$$

From this representation it is immediately clear that for *linear* models, that is $\vec{\lambda}_x = \vec{\lambda}_y = \vec{\lambda}$, $\vec{\omega}_x = \vec{\omega}_y = \vec{\omega}$ and $\vec{\zeta}$ assumed to be constant, scheme (4.11) is completely symmetric with respect to t and, therefore, second order accurate with respect to the time step τ. For *non-linear* models we have at internal grid points a truncation error given by (substitution of a solution of (1.1) into (4.11) and calculation of the residual term left)

$$\tfrac{1}{2}\tau[V\frac{\partial\omega_x}{\partial t} - U\frac{\partial\lambda_x}{\partial t}] + g([\frac{\partial}{\partial x}] - \frac{\partial}{\partial x})Z + O(\tau^2) ,$$

$$(4.12) \qquad -\tfrac{1}{2}\tau[U\frac{\partial\omega_y}{\partial t} + V\frac{\partial\lambda_y}{\partial t}] + g([\frac{\partial}{\partial y}] - \frac{\partial}{\partial y})Z + O(\tau^2),$$

$$-\tfrac{1}{2}\tau\Big[[\frac{\partial}{\partial x}](U\frac{\partial\zeta}{\partial t}) + [\frac{\partial}{\partial y}](V\frac{\partial\zeta}{\partial t})\Big] + ([\frac{\partial}{\partial x}] - \frac{\partial}{\partial x})(\zeta U) + ([\frac{\partial}{\partial y}] - \frac{\partial}{\partial y})(\zeta V) + O(\tau^2)$$

for the U, V and Z equations, respectively.

5. STABILITY ANALYSIS

5.1. THE AMPLIFICATION MATRIX

Following RICHTMYER & MORTON [1967] we replace in the single step difference scheme

$$(5.1) \qquad \vec{s}_{n+1} = A_n \vec{s}_n + \vec{b}_n$$

the difference operators $[\frac{\partial}{\partial X}]$ and $[\frac{\partial}{\partial Y}]$ by scalars δ_X and δ_Y defined by

$$(5.2) \qquad \begin{aligned} &[\tfrac{\partial}{\partial X}]\exp[i\Delta(j\gamma_1+k\gamma_2)] = \delta_X[\exp[\Delta(j\gamma_1+k\gamma_2)]] \\ &[\tfrac{\partial}{\partial Y}]\exp[i\Delta(j\gamma_1+k\gamma_2)] = \delta_Y[\exp\ \Delta(j\gamma_1+k\gamma_2)] \end{aligned} ,$$

and we replace the vectors $\vec{\lambda}$, $\vec{\omega}$, etc. by the values of the components of these vectors corresponding to the grid point (x_j,y_k) (the so-called *constant coefficient method*). In this way, the matrix A becomes a (3*3) matrix which we denote by $\tilde{A}_n(\gamma_1,\gamma_2)$. This matrix will be called the *amplification matrix* at (x_j,y_k). For instance at internal grid points the Euler scheme (4.3) has the amplification matrix

$$(5.3) \qquad \tilde{A}_n(\gamma_1,\gamma_2) = I + \tau_n \tilde{D}_n(\gamma_1,\gamma_2),$$

where

$$(5.4) \qquad \tilde{D}_n(\gamma_1,\gamma_2) = - \begin{pmatrix} \lambda_x & -\omega_x & \alpha g \\ \omega_y & \lambda_y & \beta g \\ \alpha\zeta & \beta\zeta & 0 \end{pmatrix} ,$$

$$\alpha = \frac{\partial X}{\partial x}\delta_X + \frac{\partial Y}{\partial x}\delta_Y , \quad \beta = \frac{\partial X}{\partial y}\delta_X + \frac{\partial Y}{\partial y}\delta_Y ,$$

We will call a difference scheme *stable* when the amplification matrices $\tilde{A}_n(\gamma_1,\gamma_2)$ have eigenvalues within or on the unit circle, provided that the eigenvalues on the unit circle have multiplicity one. This condition on the eigenvalues of $\tilde{A}_n(\gamma_1,\gamma_2)$ should be satisfied for all n, all grid points (x_j,y_k) and all (γ_1,γ_2). Furthermore, we will call a difference scheme *strongly stable* when all eigenvalues are within the unit circle.

5.2. THE EULER AND RUNGE-KUTTA SCHEMES

When the dependence on n of $\tilde{D}_n$ is ignored, we can express the eigenvalues of the amplification matrix of the Euler and Runge-Kutta schemes in terms of those of $\tilde{D}_n$. Denoting the eigenvalues of $\tilde{D}_n$ and $\tilde{A}_n$ by δ and μ, respectively, we have

$$(5.5) \qquad \mu_{\text{Euler}} = 1 + \tau_n\delta \;, \qquad \mu_{\text{Runge-Kutta}} = 1 + \tau_n\delta + \tfrac{1}{2}\tau_n^2\delta^2 + \tfrac{1}{4}\tau_n^3\delta^3 \;,$$

where δ satisfies the cubic equation

$$(5.6) \qquad \delta(\delta+\lambda_x)(\delta+\lambda_y) - g\zeta[\alpha^2(\delta+\lambda_y) + \beta^2(\delta+\lambda_x)] + \\ + \omega_x\omega_y\delta + g\zeta\alpha\beta(\omega_y-\omega_x) = 0.$$

It can be proved (cf. VAN DER HOUWEN [1968]) that the Euler scheme (4.3) and Runge-Kutta scheme (4.4) are strongly stable when $(|\mu|<1)$

$$(5.7) \qquad |\tau_n\delta + 1| < 1 \quad \text{and} \quad |\tau_n\delta| < 1.72,\ \mathrm{Re}\,\tau_n\delta \le 0,\ \tau_n\delta \ne 0$$

respectively.

In order to get an impression of the location of the eigenvalues δ, we consider (5.6) for $\omega_x = \omega_y = 0$, $\lambda_x = \lambda_y = \lambda$ (no Coriolis force and convective inertia terms) and for $\omega_x = \omega_y = \omega$, $\lambda_x = \lambda_y = 0$ (no bottom friction and convective intertia terms), respectively. In the first case we find

$$(5.8) \qquad \delta_1 = -\lambda, \quad \delta_{2,3} = -\tfrac{1}{2}\lambda \pm \sqrt{(\tfrac{\lambda}{2})^2 + g\zeta(\alpha^2+\beta^2)},$$

and in the second case

$$(5.9) \qquad \delta_1 = 0, \qquad \delta_{2,3} = \pm \sqrt{-\omega^2 + g\zeta(\alpha^2+\beta^2)}.$$

It is easily verified that α and β assume purely imaginary values for every symmetric discretization of $\partial/\partial X$ and $\partial/\partial Y$. For instance, when $[\partial/\partial X]$ is defined by

$$(5.10) \qquad [\frac{\partial}{\partial X}]f(j\Delta,k\Delta) = \frac{1}{4\Delta}[f((j+1)\Delta,(k+1)\Delta) - f((j-1)\Delta,(k+1)\Delta)]$$

$$+ \frac{1}{4\Delta}[f((j+1)\Delta,(k-1)\Delta) - f((j-1)\Delta,(k-1)\Delta)]$$

and $[\partial/\partial Y]$ in a similar way, then it follows from (5.2) that

$$(5.11) \qquad \delta_X = i\,\frac{\sin\gamma_1\Delta\cos\gamma_2\Delta}{\Delta}, \quad \delta_Y = i\,\frac{\sin\gamma_2\Delta\cos\gamma_1\Delta}{\Delta}\ .$$

Hence, α and β also are purely imaginary. From (5.8) and (5.9) it may now be concluded that the eigenvalues of $\tilde{D}_n$ are purely imaginary for zero friction and zero inertia terms, while for zero Coriolis force and zero inertia terms the eigenvalues are forced into the left half plane.

Returning to the eigenvalues of the amplification matrix of the Euler scheme, we see from (5.7) that the eigenvalues δ should be in the left half plane, otherwise no stability is obtained. An analysis of LAUWERIER & DAMSTÉ [1961] and VAN DER HOUWEN [1968] revealed that for realistic values of the bottom friction λ, the eigenvalues δ remain so close to the imaginary axis that unrealistically small time steps are required to satisfy (5.7).

In case of the Runge-Kutta scheme (4.4), condition (5.7) is satisfied when

$$(5.12) \qquad \tau_n \quad \frac{1.72}{|\delta|_{max}}\ .$$

5.3. GENERALIZED EULER SCHEMES

The amplification matrix of scheme (4.5) is given by

(5.13) $$\tilde{A}_n = I + \tau_n \tilde{\Theta}_n \tilde{D}_n \ .$$

The eigenvalues μ of $\tilde{A}_n$ are the roots of the equation

(5.14) $$\det[\tilde{\Theta}_n] \cdot \det[(1-\mu)\Theta_n^{-1} + \tau_n \tilde{D}_n] = 0.$$

We shall consider this equation for the symmetrized scheme (4.10). Assuming that Θ_n is non-singular, we obtain

(5.14') $$\det \begin{bmatrix} (1-\mu)(1+\tfrac{1}{2}\tau\lambda_x) - \tau\lambda_x & -\tfrac{1}{2}\tau\omega_x(1-\mu) + \tau\omega_x & -\tau\alpha g \\ \tfrac{1}{2}\tau\omega_y(1-\mu) - \tau\omega_y & (1-\mu)(1+\tfrac{1}{2}\tau\lambda_y) - \tau\lambda_y & -\tau\beta g \\ \tau\alpha\zeta(1-\mu) - \tau\alpha\zeta & \tau\beta\zeta(1-\mu) - \tau\beta\zeta & 1-\mu \end{bmatrix} = 0.$$

or equivalently

(5.14") $$\begin{aligned} & b_0(\mu-1)^3 + b_1(\mu-1)^2 + b_2(\mu-1) + b_3 = 0, \\ & b_0 = (1+\frac{\tau}{2}\lambda_x)(1+\frac{\tau}{2}\lambda_y) + \frac{\tau^2}{4}\omega_x\omega_y \ , \\ & b_1 = +\tau(\lambda_x+\lambda_y) + b_2 - b_3, \\ & b_2 = \tau^2(\lambda_x\lambda_y + \omega_x\omega_y) - \tau^2 g\zeta(\alpha^2+\beta^2) + \frac{3}{2}b_3, \\ & b_3 = -\tau^3 g\zeta(\lambda_y\alpha^2 + (\omega_x-\omega_y)\alpha\beta + \lambda_x\beta^2). \end{aligned}$$

The roots of (5.14") are within the unit circle when the following inequalities are satisfied (cf. VAN DER HOUWEN [1968]

(5.15) $$\begin{aligned} & b_0 > 0, \\ & b_3 > 0, \\ & 8b_0 - 4b_1 + 2b_2 - b_3 > 0, \\ & 2b_2 - 3b_3 > 0, \\ & (b_2-b_3)(b_1-b_2+b_3) > 0. \end{aligned}$$

A simple calculation yields the inequalities.

$$(5.15') \quad \begin{aligned} &1 + \tfrac{1}{2}\tau(\lambda_x+\lambda_y) + \tfrac{1}{4}\tau^2(\lambda_x\lambda_y + \omega_x\omega_y) > 0, \\ &\lambda_y\alpha^2 + (\omega_x-\omega_y)\alpha\beta + \lambda_x\beta^2 < 0, \\ &4 + \tau^2 g\zeta(\alpha^2+\beta^2) > 0, \\ &\lambda_x\lambda_y + \omega_x\omega_y - g\zeta(\alpha^2+\beta^2) > 0, \\ &\lambda_x + \lambda_y > 0. \end{aligned}$$

These inequalities are satisfied when the time step satisfies the condition

$$(5.16) \quad \tau < \frac{2}{\sqrt{-g\zeta(\alpha^2+\beta^2)}}, \quad \alpha^2 + \beta^2 \neq 0$$

and when $\lambda_x, \lambda_y, \omega_x$ and ω_y satisfy the inequalities

$$(5.17) \quad \begin{aligned} &\lambda_x > 0 \quad , \lambda_y > 0, \\ &(\lambda_x-\lambda_y)^2 < 4\omega_x\omega_y, \\ &(\omega_x-\omega_y)^2 < 4\lambda_x\lambda_y. \end{aligned}$$

For $\alpha = \beta = 0$ equation (5.14") reduces to

$$(5.18) \quad (\mu-1)[b_0(\mu-1)^2 + b_1(\mu-1) + b_2] = 0,$$

hence one root lies on the unit circle; the other ones can be forced within the unit circle by requiring that

$$(5.19) \quad \begin{aligned} &b_0 > 0, \\ &b_2 - b_1 < 0, \\ &b_2 - b_0 > 0, \\ &3b_0 - 2b_1 + b_2 > 0. \end{aligned}$$

These conditions are satisfied when (5.17) holds.

We observe that conditions (5.17) are trivially fulfilled for models *without* convective inertia terms but *with* bottom friction. For such models

BIBLIOTHEEK MATHEMATISCH CENTRUM
AMSTERDAM

the symmetrized scheme is strongly stable when we ignore the limiting case $\alpha = \beta = 0$.

5.4. ESTIMATION OF THE GRID PARAMETER ρ

We define the parameter

$$(5.20)\qquad \rho = \frac{1}{\sqrt{|\alpha^2+\beta^2|_{max}}} \;.$$

From the definition of α and β (cf. formula (5.4)) it follows that ρ is completely determined by the grid in the (x,y)-plane. Furthermore, by considering the stability conditions derived in the preceding sections, we see that the maximal time step is approximately proportional to ρ.

In this section we shall derive a *lower bound* for ρ in the case of average central differences. From the definition of α and β and the expressions (2.6) and (2.7) for the partial derivatives of X and Y it follows that

$$(5.21)\qquad \alpha^2 + \beta^2 = \frac{4\Delta^2}{\sin^2(\alpha_2-\alpha_1)} \left\{\left[\frac{\sin\alpha_2}{2\bar{\xi}}\,\delta_X + \frac{\sin\alpha_1}{2\bar{\eta}}\,\delta_Y\right]^2 + \left[\frac{\cos\alpha_2}{2\bar{\xi}}\,\delta_X + \frac{\cos\alpha_1}{2\bar{\eta}}\,\delta_Y\right]^2\right\} =$$

$$= \frac{4\Delta^2}{\sin^2(\alpha_2-\alpha_1)} \left\{\left(\frac{\delta_X}{2\bar{\xi}}\right)^2 + \frac{\delta_X\delta_Y\cos(\alpha_2-\alpha_1)}{2\bar{\xi}\bar{\eta}} + \left(\frac{\delta_Y}{2\bar{\eta}}\right)^2\right\},$$

where

$$\bar{\xi} = \frac{\xi_1+\xi_2}{2}, \quad \bar{\eta} = \frac{\eta_1+\eta_2}{2} \;.$$

It can be proved (cf. VAN DER HOUWEN [1968]) that in coase of (5.11)

$$\left(\frac{\delta_X}{\bar{\xi}}\right)^2 + \left(\frac{\delta_Y}{\bar{\eta}}\right)^2 > -\frac{1}{\Delta^2}\,[\min(\bar{\xi},\bar{\eta})]^{-2}.$$

Furthermore, we have

$$\delta_X \delta_Y \geq - \frac{1}{4\Delta^2} .$$

Thus,

(5.22) $$\alpha^2 + \beta^2 \geq - \sin^{-2}(\alpha_2-\alpha_1)\{[\min(\bar{\xi},\bar{\eta})]^{-2} + \tfrac{1}{2}(\bar{\xi}\bar{\eta})^{-1}|\cos(\alpha_2-\alpha_1)|\}.$$

For ρ we find an approximation from below, given by

(5.23) $$\rho \cong \min(\bar{\xi},\bar{\eta}) \sqrt{\frac{2\bar{\xi}\bar{\eta} \sin^2(\alpha_2-\alpha_1)}{2\bar{\xi}\bar{\eta}+[\min(\bar{\xi},\bar{\eta})]^2|\cos(\alpha_2-\alpha_1)|}} .$$

As already observed the maximal step length allowed by the stability condition increases when ρ increases. Therefore, it is of interest how ρ changes with $\alpha_2 - \alpha_1$ for given $\bar{\xi}$ and $\bar{\eta}$. In figure 5.1 the behaviour of ρ as a function of $\alpha_2 - \alpha_1$ is illustrated for $\bar{\xi} = \bar{\eta}$, i.e.

(5.23') $$\rho \cong \bar{\xi} \sqrt{\frac{2 \sin^2(\alpha_2-\alpha_1)}{2+|\cos(\alpha_2-\alpha_1)|}} .$$

From this we may conclude that for grids where the grid lines are making angles not less than 60°, only 20% of the maximum value of ρ is lost. Smaller angles give rise to increasingly smaller values of ρ. A similar argument holds for the case $\bar{\xi} \neq \bar{\eta}$.

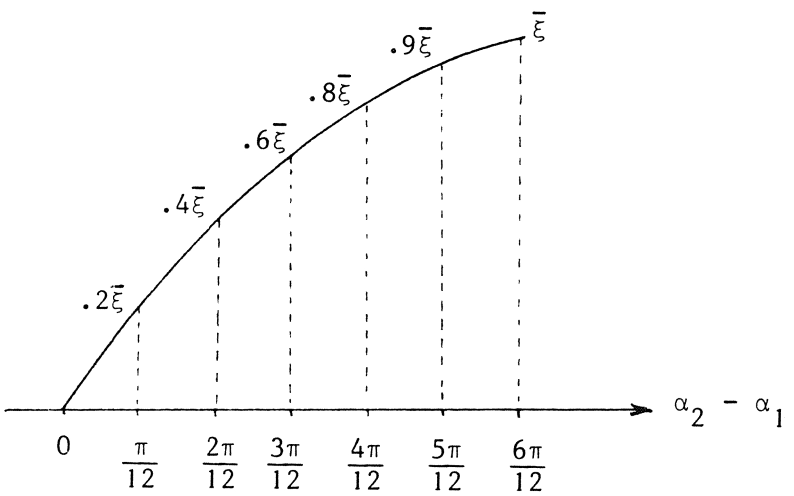

Fig. 5.1 Behaviour of the gridparameter as a function of $\alpha_2 - \alpha_1$

5.5. SURVEY OF STABILITY CONDITIONS

In this section the stability conditions derived for the various shallow water schemes are collected. In order to compare them, these conditions are expressed in terms of ρ, and convective inertia terms are neglected. Moreover, we list the maximal time step allowed by the stability condition when the North Sea conditions are substituted, i.e.

$$\lambda \cong 25_{10}{-6}\ \text{sec}^{-1},\quad \omega \cong 125_{10}{-6}\ \text{sec}^{-1},\quad g \cong 10\ \text{msec}^{-2}$$

Lauwerier-Damstê (4.3)

$$\tau \le \left\{\frac{\lambda\rho^2}{g\zeta}, \frac{2}{3\lambda}, \frac{2\lambda}{\lambda^2+\omega^2}\right\}_{\min} \cong$$

$$\cong \left\{25_{10}{-7}\ \frac{\rho^2}{\zeta},\ 3000\right\}_{\min}\ \text{sec}$$

Stabilized Runge-Kutta (4.4) $\tau \le \dfrac{1.72\rho}{\sqrt{g\zeta}} \cong .56\sqrt{\dfrac{\rho^2}{\zeta}}$ sec

Fischer-Hansen (4.6)

$$\tau \le \left\{\frac{2\rho\sqrt{1-\frac{1}{2}\tau\lambda}}{\sqrt{g\zeta}}, \frac{\lambda}{\lambda^2+\omega^2}\right\}_{\min} \cong$$

$$\cong \left\{.63\sqrt{\frac{\rho^2}{\zeta}},\ 1500\right\}_{\min}\ \text{sec}$$

Sielecki (4.7)

$$\tau \le \frac{2\rho}{\sqrt{g\zeta+\omega^2\rho^2}} \cong .63\sqrt{\frac{\rho^2}{\zeta}}\ \text{sec}$$

Leendertse (4.9)

$$\tau \le \frac{2}{\omega} \cong 16000\ \text{sec}$$

Symmetrized scheme (4.10)

$$\tau \le .63\sqrt{\frac{\rho^2}{g\zeta}} \cong .63\sqrt{\frac{\rho^2}{\zeta}}\ \text{sec}$$

We recall that these stability conditions are to be satisfied at every grid point (x_j, y_k).

6. CHOICE OF MESH SIZES AND TIME STEPS IN EXPLICIT SCHEMES

Let us investigate the consequences of the stability conditions for explicit difference schemes. In particular, we will consider the symmetrized scheme (4.10). From section 5.5 it follows that for a prescribed time step τ, the grid should be chosen such that the grid parameter ρ satisfies the inequality

$$(6.1) \qquad \rho \geq \tfrac{1}{2}\tau\sqrt{g\zeta}.$$

For instance, when we have a grid in which the grid lines make angles $(\alpha_2-\alpha_1)$ not less than 60^0 (cf. figure 2.1), then by virtue of (5.23) we find

$$\rho \geq \sqrt{\frac{3}{5}} \min\,(\bar{\xi},\bar{\eta}),$$

hence condition (6.1) is certainly satisfied when the mesh sizes in the (x,y)-plane satisfy the inequality

$$(6.1') \qquad \min(\bar{\xi},\bar{\eta}) \geq \frac{1}{2}\sqrt{\frac{5}{3}}\,\tau\sqrt{g\zeta} \cong 2\tau\sqrt{\zeta}.$$

When a mesh size strategy is used based on (6.1), we have large meshes in the deep sea areas and small meshes in the shallow parts of the sea. When the shallow regions are located along the coast and the deeper parts far from the coast, the above strategy for choosing meshes in the (x,y)-plane seems to be reasonable for predicting water elevations along the coast. However, when the depth function h is rapidly increasing when moving from the coast, the coarse meshes are too close to the coast and may influence the accuracy of the results at the coast. In order to overcome this difficulty, one may use more accurate discretizations of $\partial/\partial y$ at points of large depths. We shall derive a fourth order approximation to $\partial/\partial X$ and $\partial/\partial Y$ and investigate the effect on the stability condition, that is we shall derive the adjusted grid parameter ρ.

6.1. FOURTH ORDER APPROXIMATIONS TO $\partial/\partial X$ AND $\partial/\partial Y$

In order to obtain higher order approximations to $\partial/\partial x$ and $\partial/\partial y$ than obtained by replacing $\partial/\partial X$ and $\partial/\partial Y$ with the usual central or averaged central differences, we have to take more grid points into account. We shall derive a fourth order approximation holding for the Lauwerier-Damsté grid (see figure 3.3). By considerations of symmetry, we start with (compare (5.10))

$$(6.2)\qquad [\tfrac{\partial}{\partial X}]f(j\Delta,k\Delta) = \frac{a}{\Delta}[f((j+3)\Delta,(k+1)\Delta) + f((j+3)\Delta,(k-1)\Delta) - f((j-3)\Delta,(k+1)\Delta) - f((j-3)\Delta,(k-1)\Delta)]$$
$$+ \frac{b}{\Delta}[f((j+1)\Delta,(k+1)\Delta) + f((j+1)\Delta,(k-1)\Delta) - f((j-1)\Delta,(k+1)\Delta) - f((j-1)\Delta,(k-1)\Delta)]$$
$$+ \frac{c}{\Delta}[f((j+1)\Delta,(k+3)\Delta) + f((j+1)\Delta,(k-3)\Delta) - f((j-1)\Delta,(k+3)\Delta) - f((j-1)\Delta,(k-3)\Delta)],$$

or more compactly

$$(6.2')\qquad \Delta[\tfrac{\partial}{\partial x}] = \begin{bmatrix} 0 & -c & c & 0 \\ -a & -b & b & a \\ -a & -b & b & a \\ 0 & -c & c & 0 \end{bmatrix},$$

and a similar expression for $[\partial/\partial Y]$.
By expanding the right hand side of (6.2) in a Taylor series at the point $(j\Delta,k\Delta)$, we can derive that

$$(6.3)\qquad [\tfrac{\partial}{\partial X}] = \frac{\partial}{\partial X} + O(\Delta^4)$$

when

$$(6.4)\qquad a = -\frac{1}{96},\quad b = \frac{30}{96},\quad c = -\frac{3}{96}.$$

By taking minus the transpose of the right hand side of (6.2'), we obtain the formula for $\Delta[\frac{\partial}{\partial Y}]$.

In order to compute the grid parameter ρ corresponding to (6.2), we need the expressions for δ_X and δ_Y as defined by (5.2). From (6.2) we have

$$\Delta[\frac{\partial}{\partial X}]\exp[(j\gamma_1+k\gamma_2)i\Delta] = \exp[(j\gamma_1+k\gamma_2)i\Delta].$$
$$\cdot\{a(e^{(3\gamma_1+\gamma_2)i\Delta}+e^{(3\gamma_1-\gamma_2)i\Delta}-e^{-(3\gamma_1-\gamma_2)i\Delta}-e^{-(3\gamma_1+\gamma_2)i\Delta}) +$$
$$b(e^{(\gamma_1+\gamma_2)i\Delta}+e^{(\gamma_1-\gamma_2)i\Delta}-e^{-(\gamma_1-\gamma_2)i\Delta}-e^{-(\gamma_1+\gamma_2)i\Delta}) +$$
$$c(e^{(\gamma_1+3\gamma_2)i\Delta}+e^{(\gamma_1-3\gamma_2)i\Delta}-e^{-(\gamma_1-3\gamma_2)i\Delta}-e^{-(\gamma_1+3\gamma_2)i\Delta})\}.$$

Thus,

$$\delta_X = \frac{1}{\Delta}\{a(e^{3i\gamma_1\Delta}-e^{-3i\gamma_1\Delta})(e^{i\gamma_2\Delta}+e^{-i\gamma_2\Delta}) +$$
$$b(e^{i\gamma_1\Delta}-e^{-i\gamma_1\Delta})(e^{i\gamma_2\Delta}+e^{-i\gamma_2\Delta}) +$$
$$c(e^{i\gamma_1\Delta}-e^{-i\gamma_1\Delta})(e^{3i\gamma_2\Delta}+e^{-3i\gamma_2\Delta})\},$$

and a similar expression for δ_Y. By some elementary operations and by substituting (6.4), we may derive

$$\delta_X = \frac{1}{6\Delta}\, i \sin\gamma_1\Delta \cos\gamma_2\Delta(6+\sin^2\gamma_1\Delta-3\cos^2\gamma_2\Delta)$$

(6.5)

$$\delta_Y = \frac{1}{6\Delta}\, i \sin\gamma_2\Delta \cos\gamma_1\Delta(6+\sin^2\gamma_2\Delta-3\cos^2\gamma_1\Delta).$$

Analogous to the derivation given in section 5.4 we derive a lower bound for $(\delta_X/\bar{\xi})^2 + (\delta_Y/\bar{\eta})^2$ and $\delta_X\delta_Y$. Let us write

$$p = \sin^2\gamma_1\Delta, \quad q = \cos^2\gamma_2\Delta.$$

Then, the following expression is obtained for $(\delta_X/\bar{\xi})^2 + (\delta_Y/\bar{\eta})^2$:

$$(6.6) \qquad (\frac{\delta_X}{\bar{\xi}})^2 + (\frac{\delta_Y}{\bar{\eta}})^2 = \frac{-1}{36\Delta^2}\left[\frac{pq(6+p-3q)^2}{\bar{\xi}^2} + \frac{(1-p)(1-q)(4+3p-q)^2}{\bar{\eta}^2}\right].$$

We investigate the right hand side for fixed values of $q(0 \leq q \leq 1)$. To that end we rewrite (6.6) in the form

$$(6.6') \qquad -\left(\frac{\delta_X}{\bar{\xi}}\right)^2 - \left(\frac{\delta_Y}{\bar{\eta}}\right)^2 = c_1\, p\, \frac{(6+p-3q)^2}{(7-3q)^2} + c_2(1-p)\frac{(4+3p-q)^2}{(4-q)^2}$$

where c_1 and c_2 are non-negative constants given by

$$c_1 = \frac{q(7-3q)^2}{36\Delta^2\bar{\xi}^2}\;, \quad c_2 = \frac{(1-q)(4-q)^2}{36\Delta^2\bar{\eta}^2}\;.$$

In figure 6.1 the behaviour of expression (6.6') is illustrated as a function of p.

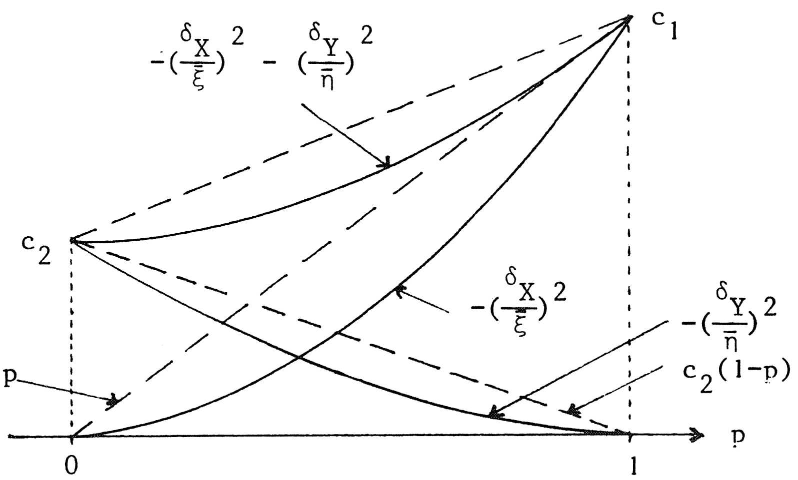

fig. 6.1 Behaviour of $-\left(\frac{\delta_X}{\bar{\xi}}\right)^2 - \left(\frac{\delta_Y}{\bar{\eta}}\right)^2$
for a fixed value of q

By considering the behaviour of the individual terms in (6.6') it is readily seen that for each value of q, $0 \leq q \leq 1$, expression (6.6') assumes its maximal value either at $p = 0$ or at $p = 1$ $(0 \leq p \leq 1)$. Consequently, we have

$$\left(\frac{\delta_X}{\bar{\xi}}\right)^2 + \left(\frac{\delta_Y}{\bar{\eta}}\right)^2 \geq \min\left\{\frac{-q(7-3q)^2}{36\Delta^2\bar{\xi}^2}\;,\; \frac{-(1-q)(4-q)^2}{36\Delta^2\bar{\eta}^2}\right\}\;.$$

A simple calculation yields

(6.7) $$\left(\frac{\delta_X}{\bar{\xi}}\right)^2 + \left(\frac{\delta_Y}{\bar{\eta}}\right)^2 \geq -\frac{343}{729\Delta^2}[\min(\bar{\xi},\bar{\eta})]^{-2}.$$

For $\delta_X\delta_Y$ we may write

(6.8) $$\delta_X\delta_Y = \frac{-1}{144\Delta^2}\sin 2\gamma_1\Delta \sin 2\gamma_2\Delta\left(\frac{10-\cos 2\gamma_1\Delta-3\cos 2\gamma_2\Delta}{2}\right)\cdot$$

$$\cdot\left(\frac{10-\cos 2\gamma_2\Delta-3\cos 2\gamma_1\Delta}{2}\right) =$$

$$= \frac{-1}{576\Delta^2}(10 \sin 2\gamma_1\Delta-\tfrac{1}{2}\sin 4\gamma_1\Delta- 3 \sin 2\gamma_1\Delta \cos 2\gamma_2\Delta)$$

$$(10 \sin 2\gamma_2\Delta-\tfrac{1}{2}\sin 4\gamma_2\Delta-3 \sin 2\gamma_2\Delta \cos 2\gamma_1\Delta)$$

$$> \frac{-1}{576\Delta^2}[100 \sin 2\gamma_1\Delta \sin 2\gamma_2\Delta-20 \sin 4\gamma_1\Delta \sin 2\gamma_2\Delta +$$

$$-20 \sin 2\gamma_1\Delta \sin 4\gamma_2\Delta + \frac{11}{2}].$$

It is easily verified that this last expression assumes its minimum at the points defined by

$$\sin 2\gamma_1\Delta = \sin 2\gamma_2\Delta = 1,$$

hence

$$\delta_X\delta_Y > -\frac{211}{1152\Delta^2}.$$

From (5.21) it now follows that

$$\alpha^2 + \beta^2 > -\sin^{-2}(\alpha_2-\alpha_1)\left\{\frac{343}{729}[\min(\bar{\xi},\bar{\eta})]^{-2} + \frac{211}{576}(\bar{\xi}\bar{\eta})^{-1}|\cos(\alpha_2-\alpha_1)|\right\}.$$

Thus, by (5.20), we obtain the following approximation from below for the grid parameter ρ.

(6.9) $$\rho \cong 1.45 \min(\bar{\xi},\bar{\eta})\sqrt{\frac{\bar{\xi}\bar{\eta}\sin^2(\alpha_2-\alpha_1)}{\bar{\xi}\bar{\eta}+.82[\min(\bar{\xi},\bar{\eta})]^2|\cos(\alpha_2-\alpha_1)|}}\ .$$

A comparison with (5.23) reveals that the 12-point formula (6.2) has a larger grid parameter, and therefore, allows larger time steps (or smaller meshes) than the usual 4-point formula.

REFERENCES

DRONKERS, J.J. [1964]: *Tidal computations in rivers and coastal waters*, John Wiley & Sons, Inc., New York.

FISCHER, G. [1959]: *Ein numerisches Verfahren zur Errechnung von Windstau und Gezeiten in Randmeeren*, Tellus, 11, 60-76.

HANSEN, W. [1956]: *Theorie zur Errechnung des Wasserstandes und der Strömungen in Randmeeren nebst Anwendungen*, Tellus, 8, 289-300.

——— [1961]: *Hydrodynamical Methods applied to oceanographic problems*, Proceedings of the symposium on Mathematical-hydrodynamical methods of physical oceanography, Institut für Meereskunde der Universität Hamburg, 1962.

HEAPS, N.S. [1969]: *A two-dimensional numerical sea model*, Phil. Trans., A 257.

HOUWEN, P.J. VAN DER [1966]: *On the stability of a difference scheme for the North Sea Problem*, Report TW 100, Mathematisch Centrum, Amsterdam.

——— [1968]: *Finite difference methods for solving partial differential equations*, MC Tract 20, Mathematisch Centrum, Amsterdam.

——— [1971]: *Stabilized Runge-Kutta methods with limited storage requirements*, Report TW 124, Mathematisch Centrum, Amsterdam.

LAUWERIER, H.A. and B.R. DAMSTÉ [1963]: *The North Sea Problem VIII, A numerical treatment*, Proc. Kon. Ned. Ak. v. Wetensch., A 66, 167-184.

PLATZMAN, G.W. [1959]: *A numerical computation of the surge of 26 June 1954 on Lake Michigan*, Geophysics, 6, 407-438.

PROUDMAN, J. [1953]: *Dynamical oceanography*, John Wiley & Sons, Inc., New York.

RICHTMYER, R.D. and K.W. MORTON [1967]: *Difference methods for initial-value problems*, Interscience Publishers, New York.

SIELECKI, A. [1968]: Mon. Weath. Rev. U.S. Dep. Agric., 96, 150-156.

ONTVANGEN 2 9 SEP. 1975